

Deriving Abbe's Certainty from Heisenberg's Uncertainties

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More than a century ago, Ernst Abbe realized that resolving power is subject to fundamental physical and not technical limits [1]. Abbe based his reasoning on the behavior of a plane monochromatic wave of wavelength λ , which is diffracted by a grating whose frequency is $1/d$. Abbe established that at least half of the light diffracted into both first two orders ($m = -1, 1$) must contribute coherently to image formation if an object is to be resolved.

$$m \cdot \lambda = 2 \cdot d \cdot n \cdot \sin \alpha \Rightarrow d \geq \frac{\lambda}{2n \cdot \sin \alpha}$$

This definition seems arbitrary and is in fact one of a number of other definitions. For this reason, it is indeed useful to find a definition on a sound physical foundation. Uncertainty principles, in particular those of Heisenberg [2, 3] provide an excellent basis.

$$\Delta p_x \cdot \Delta x \geq \frac{\hbar}{2} \quad \wedge \quad \vec{p} = \hbar \cdot \vec{k} \Rightarrow \Delta k_x \cdot \Delta x \geq \frac{1}{2}$$

It is interesting to note that Δk_x and Δx are not, as it is often assumed, the maximum angles of the beam or the maximum width of the beam diameter, but the standard deviations of the respective distribution functions! The intellectually challenging step is now to stringently apply this knowledge [4]. If we use a uniform, monochromatic, spherical incident wave, we obtain an analytical term which, after a few simplifications, is very similar to Abbe's.

$$d_x \geq \frac{\lambda}{n \cdot \sqrt{3 - 2 \cos \alpha_{\max} - \cos 2\alpha_{\max}}} \xrightarrow{\text{paraxial \& cylindrical}} d \geq \frac{\lambda}{2n \cdot \sin \alpha_{\max}}$$

Whereas Abbe's formula is a paraxial approximation, the formula on the left applies to all angles, i.e. including wide numerical apertures (i.e. $\alpha_{\max} > 45^\circ$). Applying the same approach one also finds an analytical expression for the axial resolution of an optical instrument, which is also valid for large angles.

$$d_z \geq \frac{\lambda}{n \cdot (1 - \cos \alpha_{\max})}$$

In contrast to commonly used formulas $d_x = d_z$ if $\alpha_{\max} = 180^\circ$. While the results outlined here impressively underline the Abbe's ingenuity, they also show that basic physics comes up with formulas, which describe physical relationships that have an even wider range of applicability. Indeed, the above formulas have already been used to calculate, for example, the resolution of rotationally symmetric binary filters (e.g. 4Pi) for any aperture [5, 6].

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